Standardized Normal Random Variates

Gary Schurman MBE, CFA

June, 2012

There are cases in mathematical finance where we want to standardize normal random variates. Standardization transforms the mean and variance of a normal random variate such that the post-standardized mean and variance is zero and one, respectively. It is interesting to note that the correlation between two non-standardized random variates is not affected by standardization. In this pdf we will prove that (1) the standardized mean is zero, (2) the standardized variance is one and (3) correlation is unchanged.

Two Normally-Distributed Non-Standardized Random Variates

Assume that we have two normally-distributed random variates x_1 and x_2. The distribution of the random variate x_1 is...

$$x_1 \sim N[\text{mean} = \mu_1, \text{variance} = \sigma_1^2] \tag{1}$$

The distribution of the random variate x_2 is...

$$x_2 \sim N[\text{mean} = \mu_2, \text{variance} = \sigma_2^2] \tag{2}$$

The first moment of the distribution of the random variate x_1 is...

$$\mathbb{E}\Bigg[x_1\Bigg] = \mu_1 \tag{3}$$

The second moment of the distribution of the random variate x_1 is...

$$\mathbb{E}\Bigg[x_1^2\Bigg] = \sigma_1^2 + \mu_1^2 \tag{4}$$

The first moment of the distribution of the random variate x_2 is...

$$\mathbb{E}\Bigg[x_2\Bigg] = \mu_2 \tag{5}$$

The second moment of the distribution of the random variate x_2 is...

$$\mathbb{E}\Bigg[x_2^2\Bigg] = \sigma_2^2 + \mu_2^2 \tag{6}$$

The correlation of the random variate x_1 and x_2 is...

$$\rho_{x1,x2} = \frac{\mathbb{E}[x_1 x_2] - \mathbb{E}[x_1]\mathbb{E}[x_2]}{\sigma_1 \sigma_2} \tag{7}$$

Using Equations (7) we can solve for the expected value of the product of x_1 and x_2 (also uses Equations (3) and (5)) which is...

$$\begin{aligned}\mathbb{E}\Bigg[x_1 x_2\Bigg] &= \rho_{x1,x2}\,\sigma_1\sigma_2 + \mathbb{E}[x_1]\mathbb{E}[x_2] \\ &= \rho_{x1,x2}\,\sigma_1\sigma_2 + \mu_1\mu_2\end{aligned} \tag{8}$$

Standardizing Normal Random Variates

We will define z_1 to be the standardized normally-distributed random variate x_1. We standardize x_1 by subtracting the mean and dividing by the standard deviation. The equation for z_1 is...

$$z_1 = \frac{x_1 - \mu_1}{\sigma_1} \tag{9}$$

We will define z_2 to be the standardized normally-distributed random variate x_2. We standardize x_2 by subtracting the mean and dividing by the standard deviation. The equation for z_2 is...

$$z_2 = \frac{x_2 - \mu_2}{\sigma_2} \tag{10}$$

Using Equation (3) above the first moment of the distribution of the random variate z_1 is...

$$\begin{aligned}
\mathbb{E}\Big[z_1\Big] &= \mathbb{E}\Big[\frac{x_1 - \mu_1}{\sigma_1}\Big] \\
&= \frac{1}{\sigma_1}\mathbb{E}\Big[x_1 - \mu_1\Big] \\
&= \frac{1}{\sigma_1}\Big(\mathbb{E}\Big[x_1\Big] - \mathbb{E}\Big[\mu_1\Big]\Big) \\
&= \frac{1}{\sigma_1}\Big(\mu_1 - \mu_1\Big) \\
&= 0
\end{aligned} \tag{11}$$

Using Equations (3) and (4) above the second moment of the distribution of the random variate z_1 is...

$$\begin{aligned}
\mathbb{E}\Big[z_1^2\Big] &= \mathbb{E}\Big[\Big(\frac{x_1 - \mu_1}{\sigma_1}\Big)^2\Big] \\
&= \mathbb{E}\Big[\frac{x_1^2 - 2x_1\mu_1 + \mu_1^2}{\sigma_1^2}\Big] \\
&= \frac{1}{\sigma_1^2}\mathbb{E}\Big[x_1^2 - 2x_1\mu_1 + \mu_1^2\Big] \\
&= \frac{1}{\sigma_1^2}\Big(\mathbb{E}\Big[x_1^2\Big] - \mathbb{E}\Big[2x_1\mu_1\Big] + \mathbb{E}\Big[\mu_1^2\Big]\Big) \\
&= \frac{1}{\sigma_1^2}\Big(\mathbb{E}\Big[x_1^2\Big] - 2\mu_1\mathbb{E}\Big[x_1\Big] + \mu_1^2\Big) \\
&= \frac{1}{\sigma_1^2}\Big(\sigma_1^2 + \mu_1^2 - 2\mu_1^2 + \mu_1^2\Big) \\
&= \frac{1}{\sigma_1^2}\Big(\sigma_1^2\Big) \\
&= 1
\end{aligned} \tag{12}$$

Using Equation (5) and the logic of Equation (11) it can be shown that the first moment of the distribution of the random variate z_2 is...

$$\mathbb{E}\Big[z_2\Big] = 0 \tag{13}$$

Using Equations (5) and (6) and the logic of Equation (12) it can be shown that the second moment of the distribution of the random variate z_2 is...

$$\mathbb{E}\Big[z_2^2\Big] = 1 \tag{14}$$

Using Equations (9), (10), (3), (5) and (8) the expected value of the product of z_1 and z_2 is...

$$\begin{aligned}
\mathbb{E}\bigg[z_1 z_2\bigg] &= \mathbb{E}\Bigg[\bigg(\frac{x_1-\mu_1}{\sigma_1}\bigg)\bigg(\frac{x_2-\mu_2}{\sigma_2}\bigg)\Bigg] \\
&= \frac{1}{\sigma_1\sigma_2}\mathbb{E}\bigg[x_1x_2 - x_1\mu_2 - x_2\mu_1 + \mu_1\mu_2\bigg] \\
&= \frac{1}{\sigma_1\sigma_2}\Bigg(\mathbb{E}\bigg[x_1x_2\bigg] - \mu_2\mathbb{E}\bigg[x_1\bigg] - \mu_1\mathbb{E}\bigg[x_2\bigg] + \mu_1\mu_2\Bigg) \\
&= \frac{1}{\sigma_1\sigma_2}\Bigg(\rho_{x1,x2}\,\sigma_1\sigma_2 + \mu_1\mu_2 - \mu_2\mu_1 - \mu_1\mu_2 + \mu_1\mu_2\Bigg) \\
&= \frac{1}{\sigma_1\sigma_2}\Bigg(\rho_{x1,x2}\,\sigma_1\sigma_2\Bigg) \\
&= \rho_{x1,x2}
\end{aligned} \tag{15}$$

Mean, Variance and Correlation of Standardized Normal Random Variates

Using Equation (11) the mean of standardized random variate z_1 is...

$$mean = \mathbb{E}\bigg[z_1\bigg] = 0 \tag{16}$$

Using Equation (13) the mean of standardized random variate z_2 is...

$$mean = \mathbb{E}\bigg[z_2\bigg] = 0 \tag{17}$$

Using Equations (11) and (12) the variance of standardized random variate z_1 is...

$$variance = \mathbb{E}\bigg[z_1^2\bigg] - \Bigg(\mathbb{E}\bigg[z_1\bigg]\Bigg)^2 = 1 \tag{18}$$

Using Equations (13) and (14) the variance of standardized random variate z_2 is...

$$variance = \mathbb{E}\bigg[z_2^2\bigg] - \Bigg(\mathbb{E}\bigg[z_2\bigg]\Bigg)^2 = 1 \tag{19}$$

Using Equations (15), (11), (13), (18) and (19) the correlation of standardized random variates z_1 and z_2 is...

$$\begin{aligned}
\rho_{z1,z2} &= \frac{\mathbb{E}[z_1z_2] - \mathbb{E}[z_1]\mathbb{E}[z_2]}{\sqrt{variance(z_1)}\sqrt{variance(z_2)}} \\
&= \frac{\rho_{x1,x2} - (0)(0)}{(1)(1)} \\
&= \rho_{x1,x2}
\end{aligned} \tag{20}$$